TOWARDS A GOOD DEFINITION OF ALGEBRAICALLY OVERTWISTED

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Abstract. Symplectic field theory (SFT) is a collection of homology theories that provide invariants for contact manifolds. We show that vanishing of any one of either contact homology, rational SFT or (full) SFT are equivalent. We call a manifold for which these theories vanish *algebraically overtwisted*.

1. Introduction

A (coorientable) contact structure ξ on a $(2n+1)$–dimensional manifold M is a hyperplane field of the tangent bundle that can be written as the kernel of a 1–form α that satisfies the inequality $\alpha \wedge d\alpha^n \neq 0$. On closed manifolds contact structures are stable under deformations, and their equivalence classes are discrete sets. Much effort has been invested in understanding 3–dimensional contact manifolds, and a rich theory has been created. For this, many different techniques have been applied ranging from topological ones to different algebraic invariants like Heegaard Floer theories [OzSz05], or contact homology. One of the first basic properties that were discovered for 3–manifolds was the distinction between *overtwisted* and *tight* contact structures. Being overtwisted is a topological property, but it has many consequences for algebraic invariants.

The algebraic invariant under consideration is Symplectic Field Theory (SFT), introduced by Eliashberg, Givental and Hofer [EGH00]. This large formalism contains in particular several versions of contact invariants, such as contact homology. These invariants, described in Section 2, are based on the count of holomorphic curves in the symplectization of a contact manifolds and are defined for contact manifolds of any odd dimension.

It has been proved by Eliashberg and Yau [Yau06] that the contact homology of any overtwisted 3–manifold is trivial. Given that the classification of such manifolds is purely topological [Eli89], it was to be expected that also the other invariants of SFT do not provide interesting information. In this article we confirm this conjecture by a more general result for contact manifolds of any odd dimension. In fact, it follows already from purely algebraic properties that vanishing of any of the different homology theories implies that the other ones also have to be trivial.

Theorem 1. *Let (M,α) be a $(2n-1)$–dimensional closed contact manifold with a non degenerate contact form. All of the following statements are equivalent:*

(i) *The contact homology (without marked points) of (M,α) vanishes.*
(ii) *The rational SFT (without marked points) of (M,α) vanishes.*
(iii) *The SFT (without marked points) of (M,α) vanishes.*
(iv) *The contact homology with marked points of (M,α) vanishes.*
(v) *The rational SFT with marked points of (M,α) vanishes.*
(vi) *The SFT with marked points of (M,α) vanishes.*

Remark 1. Note that any of these invariants may be defined over different coefficient rings. In the theorem we assume that the same ring is used for the different homologies.

To date no final generalization of overtwisted contact manifolds to higher dimensions has been found. This theorem, together with [Yau06], motivates the following definition that can be easily applied to any dimension.

Definition. A contact manifold (M,α) is called **algebraically overtwisted**, if any of the homologies listed in Theorem 1 vanishes.

Several examples of algebraically overtwisted contact structures are known: As stated above the contact homology of overtwisted manifolds vanishes [Yau06]. Similarly Otto van Koert and the first author of this article have extended this result by showing that the contact homology of negatively stabilized contact manifolds of any dimension is trivial [BvK08].

Corollary 2. *All of these examples are thus algebraically overtwisted, and have vanishing SFT.*

A tentative generalization of the notion of overtwisted to higher dimensions was given in [Nie06], where PS–overtwisted manifolds were defined. Current work by the authors [BN] will show that PS–overtwisted manifolds also have vanishing contact homology.

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2. Contact homology and variants of SFT

The different contact invariants above are homologies of certain differential graded algebras with a $\mathbf{1}$–element, that means

Definition. A differential graded algebra $(\mathcal{A}, \partial)$ is a graded algebra, equipped with a differential $\partial : \mathcal{A}_* \to \mathcal{A}_{*-1}$ such that $\partial^2 = 0$ and which satisfies the graded Leibniz rule $\partial(a \cdot b) = (\partial a) \cdot b + (-1)^{|a|} a \cdot (\partial b)$.

The vanishing results in this article are all based on the following easy remark.

Remark 2. Let $(\mathcal{A}, \partial)$ be a differential graded algebra with $\mathbf{1}$, and denote the homology over that algebra by $H_*(\mathcal{A}, \partial)$. The homology vanishes if and only if $\mathbf{1}$ is an exact element, that means if there is an element $a \in \mathcal{A}$ such that $\partial a = \mathbf{1}$.

Proof. The $\mathbf{1}$–element is closed, because $\partial \mathbf{1} = \partial(\mathbf{1} \cdot \mathbf{1}) = (\partial \mathbf{1}) \cdot \mathbf{1} + (-1)^{|\mathbf{1}|} \mathbf{1} \cdot \partial \mathbf{1} = 2\, \partial \mathbf{1}$, hence it is obvious that $\mathbf{1}$ has to be exact for the homology to vanish. On the other hand, if there is an element $a \in \mathcal{A}$ such that $\partial a = \mathbf{1}$, then the whole homology has to vanish, because we can write an arbitrary cycle $b \in \mathcal{A}$ as $b = \mathbf{1} \cdot b = (\partial a) \cdot b = \partial(a \cdot b) - (-1)^{|a|} a \cdot \partial b = \partial(a \cdot b)$. □

Our proof of the main theorem is based on using algebraic properties and exploiting that the contact homology algebra embeds naturally into both the rational SFT algebra and the full SFT algebra. In particular the $\mathbf{1}$–elements all coincide under these inclusions. Before starting to describe the actual proof, we will briefly repeat how the algebras are defined, and what the corresponding boundary operators are (see [EGH00]). Since the algebras of the different versions of SFT are all build up by using closed Reeb orbits, and the corresponding differentials all count certain holomorphic curves, we will first fix some common notation. Readers familiar with symplectic field theory can safely skip the next section, and skim back when needed.

Notation: Closed Reeb orbits and holomorphic curves. Let α be a contact form for the $(2n-1)$–dimensional contact manifold (M, ξ). The associated Reeb vector field R_α is defined as the unique solution of the equations

$$\alpha(R_\alpha) = 1 \text{ and } i_{R_\alpha} d\alpha = 0 \,.$$

A closed Reeb orbit γ of R_α is called non degenerate, if the corresponding Poincaré return map does not have eigenvalues of size 1. We call α a non degenerate contact form, if all of its Reeb orbits are non degenerate. Any contact form can be made non degenerate by a small perturbation, and so we will always assume from now on that α is non degenerate. The associated Reeb vector field R_α then has only countably many closed orbits, and we can introduce a total order on the set of closed Reeb orbits $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots)$. Note that multiple orbits are considered to be completely unrelated to the corresponding simple orbits. Denote the period of an orbit γ by $T(\gamma)$, and its multiplicity by κ_γ. We fix a parametrization for each closed orbit γ_k by choosing a base point on γ_k. A convenient short hand notation to handle ordered tuples of closed Reeb orbits is to consider

sequences $I = (i_k)_k \in \mathbb{N}_0^{\mathbb{N}}$ with only finitely many non zero elements, and to denote by $\boldsymbol{\gamma}^I$ the tuple of orbits

$$\big(\underbrace{\gamma_1,\dots,\gamma_1}_{i_1},\dots,\underbrace{\gamma_N,\dots,\gamma_N}_{i_N}\big)\;,$$

where N is large enough to capture all non vanishing elements of I. We allow for I also the sequence $\mathbf{0} = (0,\dots)$ giving rise to the empty tuple $\boldsymbol{\gamma}^{\mathbf{0}} = ()$. Finally, let $|I|$ be the number of non zero components in the sequence I, and $C(I)$ be the integer

$$C(I) = |I|!\; i_1!\cdots i_N!\, \kappa_{\gamma_1}^{i_1}\cdots\kappa_{\gamma_N}^{i_N}$$

again for N large enough.

To compute the Conley-Zehnder index $\mathrm{CZ}(\gamma)$ of a closed Reeb orbit γ, we have to fix a trivialization of the contact structure ξ along γ. To do this in a unified way, choose a basis $A_1,\dots,A_s$ of $H_1(M,\mathbb{Z})$ (for $H_1(M,\mathbb{Z})$ with torsion, we refer to [EGH00, Section 2.9.1]) and for each element A_j a closed path ψ_j representing A_j. Fix a trivialization of ξ along ψ_j in an arbitrary way, and choose for every closed Reeb orbit γ, a surface S_γ bounding γ and the corresponding combination of ψ_j's that represent $[\gamma] \in H_1(M,\mathbb{Z})$, then use S_γ to extend the trivialization of ξ from the ψ_j's to γ.

Let $\big(\mathbb{R}\times M, d(e^t\alpha)\big)$ be the symplectization of (M,α). A complex structure J on a symplectic vector bundle (E,Ω) is called compatible with Ω, if $\Omega(J\cdot,J\cdot) = \Omega(\cdot,\cdot)$, and if $\Omega(J\cdot,\cdot)$ defines a metric. We choose a compatible $\mathbb{R}$–invariant complex structure J on the symplectic vector bundle $(\xi, d\alpha)$ and extend it to an almost complex structure on the symplectization by $J\frac{\partial}{\partial t} = R_\alpha$. To define the differentials for the different homologies, we have to enumerate certain holomorphic curves. Let (Σ_g, j) be a compact Riemann surface of genus g, and let $I^+ = (i_k^+)_k$, and $I^- = (i_l^-)_l$ be finite sequences of integers. Associate to every $i_k^+ \neq 0$ points $\overline{x}_k^1,\dots,\overline{x}_k^{i_k^+} \in \Sigma_g$, to every $i_l^- \neq 0$ points $\underline{x}_l^1,\dots,\underline{x}_l^{i_l^-} \in \Sigma_g$, together with nonzero tangent vectors $\overline{v}_k^i \in T_{\overline{x}_k^i}\Sigma_g$ and $\underline{v}_l^j \in T_{\underline{x}_l^j}\Sigma_g$ respectively. For reasons that will become clear below, we call $\overline{\mathbf{x}} = \{\overline{x}_k^l\}$ the positive, $\underline{\mathbf{x}} = \{\underline{x}_k^l\}$ the negative punctures, and the attached vectors are called asymptotic markers. Additionally let there be m marked points $y_1,\dots,y_m$ on the Riemann surface Σ_g. All the marked points and the positive and negative punctures have to be pairwise distinct.

A map

$$\tilde u = (a,u) : \big(\Sigma_g \setminus (\overline{\mathbf{x}}\cup\underline{\mathbf{x}}), j\big) \to (\mathbb{R}\times M, J)$$

is a (j,J)–holomorphic map, if $J\circ D\tilde u = D\tilde u\circ j$ for every point of $\Sigma_g\setminus(\overline{\mathbf{x}}\cup\underline{\mathbf{x}})$. Additionally we require the following properties at the punctures. Choose for every puncture $p \in \overline{\mathbf{x}}\cup\underline{\mathbf{x}}$ a holomorphic chart $D^2 \to \Sigma_g$ such the origin is mapped to p, and such that the asymptotic marker points along the positive real axis. In polar coordinates $\big(\rho e^{i\vartheta}\big) \in D^2$, the following asymptotic conditions have to be satisfied by $\tilde u$:

$$\lim_{\rho\to 0} a\big(\rho e^{i\vartheta}\big) = \begin{cases} +\infty & \text{if } p\in\overline{\mathbf{x}},\\ -\infty & \text{if } p\in\underline{\mathbf{x}},\end{cases} \qquad\text{and}\qquad \lim_{\rho\to 0} u\big(\rho e^{i\vartheta}\big) = \begin{cases} \gamma_k(-\frac{T_k}{2\pi}\vartheta) & \text{if } p = \overline{x}_k^i,\\ \gamma_l(\frac{T_l}{2\pi}\vartheta) & \text{if } p = \underline{x}_l^j,\end{cases}$$

where T_k denotes the period of the orbit γ_k. When we do not want to fix the complex structure on Σ_g, we call such a map a J–holomorphic map.

Choose an additional puncture x_0 with asymptotic marker that will be asymptotic to a closed Reeb orbit γ. We denote by $\mathcal{M}_{g,m}^A\big(\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+},\gamma\big)$ the space of J–holomorphic maps as above that have an additional positive puncture x_0, and by $\mathcal{M}_{g,m}^A\big(\gamma,\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+}\big)$ the space of J–holomorphic maps that have an additional negative puncture x_0. In both cases, we assume that the surface obtained by gluing $u(\Sigma_g\setminus(\{x_0\}\cup\overline{\mathbf{x}}\cup\underline{\mathbf{x}}))$ with suitable surfaces S_{γ_k} represents the homology class $A \in H_2(M,\mathbb{Z})$.

Let (Σ_g, j) and (Σ_g', j') be compact Riemann surfaces equipped with positive and negative punctures $\overline{\mathbf{x}},\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}',\underline{\mathbf{x}}'$ respectively and with m marked points $y_1,\dots,y_m$ and $y_1',\dots,y_m'$. We call a diffeomorphism $\varphi:\Sigma_g\to\Sigma_g'$ a reparametrization, if it is a biholomorphism that is compatible with all special points. This means that φ satisfies the equation $\varphi_*j = j'$, and $\varphi(y_k) = y_k'$, and corresponding relations for the ordered punctures. The map also has to respect the asymptotic

markers at each puncture. Define an equivalence relation $\sim$ on the space of maps $\mathcal{M}^A_{g,m}(\dots)$ by saying that two maps $\tilde u = (a,u)$ and $\tilde u' = (a',u')$ are equivalent, if there is a shift $\tau \in \mathbb{R}$, and a reparametrization $\varphi\colon (\Sigma_g, j) \to (\Sigma_g, j')$ such that

$$(a,u) = (a' \circ \varphi + \tau, u' \circ \varphi)\;.$$

The moduli spaces

$$\widehat{\mathcal{M}}^A_{g,m}(\dots) = \mathcal{M}^A_{g,m}(\dots)/\sim$$

are obtained by dividing out the corresponding space of maps by the equivalence relation $\tilde u \sim \tilde u'$ just defined. Denote the first Chern class of the complex vector bundle $(\xi = \ker\alpha, J)$ by $c_1(\xi)$. If the elements of $\widehat{\mathcal{M}}^A_{g,m}(\dots)$ are not branched coverings, then choosing J generically these moduli spaces are smooth orbifolds of dimension

$$\begin{aligned}
\dim \widehat{\mathcal{M}}^A_{g,m}\bigl(\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+},\gamma\bigr) &= (n-3)\,\bigl(2-2g-\left|I^-\right|-\left|I^+\right|-1\bigr)-1+2m+\mathrm{CZ}(\gamma)\\
&\quad + 2\,\langle c_1(\xi)|A\rangle + \sum_{j=1}^{\infty}(i_j^+ - i_j^-)\;\mathrm{CZ}(\gamma_j)\\
\dim \widehat{\mathcal{M}}^A_{g,m}\bigl(\gamma,\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+}\bigr) &= (n-3)\,\bigl(2-2g-1-\left|I^-\right|-\left|I^+\right|\bigr)-1+2m-\mathrm{CZ}(\gamma)\\
&\quad + 2\,\langle c_1(\xi)|A\rangle + \sum_{j=1}^{\infty}(i_j^+ - i_j^-)\;\mathrm{CZ}(\gamma_j)
\end{aligned}$$

that are equipped with a smooth evaluation map at the marked points

$$\mathrm{ev}\colon\, \widehat{\mathcal{M}}^A_{g,m}(\dots) \to M^m,\; [a,u] \mapsto \bigl(u(y_1),\dots,u(y_m)\bigr)\;.$$

The moduli spaces have compactifications $\overline{\mathcal{M}}^A_{g,m}\bigl(\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+},\gamma\bigr)$, and $\overline{\mathcal{M}}^A_{g,m}\bigl(\gamma,\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+}\bigr)$ respectively consisting of holomorphic buildings of arbitrary height [BEH+03].

In the presence of branched coverings, the new ongoing approach to transversality by Cieliebak and Mohnke (see [CM07] for the symplectic case) or the polyfold theory developed by Hofer, Wysocki and Zehnder [Hof08, HWZ07] give to the moduli space $\overline{\mathcal{M}}^A_{g,m}(\dots)$ the structure of a branched manifold (with rational weights) with boundary and corners. The presence of these rational weights is due to the use of multivalued perturbations.

In the absence of marked points, and when $\dim \widehat{\mathcal{M}}^A_{g,0}(\dots) = 0$, this moduli space consists of finitely many elements with rational weights. We denote the sum of these rational weights by $n^A_g\bigl(I^-;I^+,\gamma\bigr)$ or $n^A_g\bigl(\gamma,I^-;I^+\bigr)$. When $m \neq 0$, we define a multilinear form $n^A_{g,m}(\dots)$ on m–tuples of closed differential forms $\Theta_1,\dots,\Theta_m$ on M by the formula

$$\langle n^A_{g,m}(\dots)|(\Theta_1,\dots,\Theta_m)\rangle = \int_{\overline{\mathcal{M}}^A_{g,m}(\dots)} \mathrm{ev}^*(\Theta_1\times\dots\times\Theta_m)\;.$$

By convention, we set the multilinear form to 0, if $\sum \deg\Theta_j \neq \dim \mathcal{M}^A_{g,m}(\dots)$, and we define a 0–multivalued form $n^A_{g,0}(\dots)$ just by using the sum $n^A_g(\dots)$ of rational weights defined above.

To define the algebras, we have to find a suitable coefficient ring. For this choose a submodule

$$\mathcal{R} \le \bigl\{A \in H_2(M,\mathbb{Z})\;\big|\;\langle c_1(\xi)|A\rangle = 0\bigr\}$$

to construct the group ring $\mathbb{Q}\bigl[H_2(M,\mathbb{Z})/\mathcal{R}\bigr]$, whose elements will be written as $\sum_{j=1}^k c_j e^{A_j}$, where $c_j \in \mathbb{Q}$ and $A_j \in H_2(M,\mathbb{Z})/\mathcal{R}$. Different choices of $\mathcal{R}$ may lead to different SFT invariants. We define a grading on $\mathbb{Q}\bigl[H_2(M,\mathbb{Z})/\mathcal{R}\bigr]$ by $\bigl|c\,e^A\bigr| = -2\,\langle c_1(\xi)|A\rangle$.

Associate to every closed Reeb orbit γ the formal variables q_γ and p_γ with gradings

$$|q_\gamma| = \mathrm{CZ}(\gamma)+n-3 \quad\text{ and } |p_\gamma| = -\,\mathrm{CZ}(\gamma)+n-3$$

Given a finite sequence of integers I, we denote by $\mathbf{q}^I$ the monomial $q_{\gamma_1}^{i_1}\cdots q_{\gamma_N}^{i_N}$ and by $\mathbf{p}^I$ the monomial $p_{\gamma_1}^{i_1}\cdots p_{\gamma_N}^{i_N}$ for N large enough.

Consider a formal variable $\hbar$ with grading $|\hbar| = 2\,(n-3)$.

Contact homology. The algebra $\mathcal{A}_{CH}$ of contact homology consists of polynomials in the q_γ's with coefficients in $\mathbb{Q}\big[H_2(M,\mathbb{Z})/\mathcal{R}\big]$. Every element can be written as a finite sum

$$f = \sum_{k=1}^{K} f_k \ ,$$

where each term f_k is of the form

$$f_k = c_k\, e^{A_k} \mathbf{q}^{I_k}$$

with every $c_k\, e^{A_k} \in \mathbb{Q}\big[H_2(M,\mathbb{Z})/\mathcal{R}\big]$ and $I_k = \big(i_{j,k}\big)_j$ is a sequence of the type described above. The grading for each such monomial is given by

$$|f_k| = \sum_{j=1}^{\infty} \Big(\mathrm{CZ}(\gamma_j) + n - 3\Big)\, i_{j,k} - 2\,\langle c_1(\xi)|A_k\rangle \ .$$

The sum of monomials $f = c_1 e^A q_{\gamma_1}^{i_1}\cdots q_{\gamma_N}^{i_N}$, and $g = c_2 e^B q_{\gamma_1}^{j_1}\cdots q_{\gamma_N}^{j_N}$ (we assume N to be large enough to include all non zero terms of both sequences $I = (i_k)_k$ and $J = (j_k)_k$) is formal, and the multiplication of f and g gives

$$fg = c_1 c_2 e^{A+B} q_{\gamma_1}^{i_1}\cdots q_{\gamma_N}^{i_N} q_{\gamma_1}^{j_1}\cdots q_{\gamma_N}^{j_N} \ ,$$

where we still have to permute the q–variables to get a monomial in normal form. For this, we impose supercommutativity $q_\gamma q_{\gamma'} = (-1)^{|q_\gamma|\,|q_{\gamma'}|} q_{\gamma'} q_\gamma$. The differential on this algebra $\mathcal{A}_{CH}$ is defined by

$$\partial q_\gamma = \sum_{A,I} \frac{n_{0,0}^A\big(\boldsymbol{\gamma}^I;\gamma\big)}{C(I)}\, e^A\, \mathbf{q}^I \ ,$$

where the sum runs over all integer valued sequences I and all homology classes A. Remember that $n_{0,0}^A\big(\boldsymbol{\gamma}^I;\gamma\big)$ counts (in the sense defined above) punctured holomorphic spheres with a single positive puncture asymptotic to γ, and negative punctures asymptotic to the orbits in $\boldsymbol{\gamma}^I$. Effectively, the sum in the definition of the differential operator is finite. On one hand, the period of γ gives an upper bound for $\sum i_k T(\gamma_k)$ (see for example [BEH+03, Lemma 5.16]), so that only finitely many sequences I need to be taken into account. On the other hand, the compactness theorem for the space $\cup_A \mathcal{M}_{g,m}^A(\boldsymbol{\gamma}^I;\gamma)$ with fixed I shows that there may only be holomorphic curves for finitely many choices of A.

For products extend the differential according to the graded Leibniz rule, i.e. $\partial(fg) = (\partial f)\ g + (-1)^{|f|}\, f\ (\partial g)$.

Rational SFT. The algebra $\mathcal{A}_{rSFT}$ of rational SFT can be interpreted as a Poisson algebra with a distinguished element $\mathbf{h}$. Since we are just interested in showing that its homology vanishes, we will only describe it as a differential graded algebra. The elements of $\mathcal{A}_{rSFT}$ can be written as

$$f = \sum_{I^+} f_{I^+}(\mathbf{q})\, \mathbf{p}^{I^+} \ ,$$

where the sum runs over all finite sequences I^+ of integers, and the coefficients $f_{I^+}(\mathbf{q}) \in \mathcal{A}_{CH}$ are elements in the contact homology algebra that depend on I^+. In other words, the elements of $\mathcal{A}_{rSFT}$ are formal power series in p–variables with coefficients in the contact homology algebra $\mathcal{A}_{CH}$. The grading of a monomial is given by

$$\Big|c\, e^A \mathbf{q}^{I^-} \mathbf{p}^{I^+}\Big| = \Big|c\, e^A \mathbf{q}^{I^-}\Big| + \sum_{j=1}^{\infty} \big(n - 3 - \mathrm{CZ}(\gamma_j)\big)\, i_j^+ \ .$$

The product between variables is supercommutative

$$q_\gamma q_{\gamma'} = (-1)^{|q_\gamma||q_{\gamma'}|} q_{\gamma'} q_\gamma, \quad q_\gamma p_{\gamma'} = (-1)^{|q_\gamma||p_{\gamma'}|} p_{\gamma'} q_\gamma \quad \text{and} \quad p_\gamma p_{\gamma'} = (-1)^{|p_\gamma||p_{\gamma'}|} p_{\gamma'} p_\gamma \ .$$

The differential $d^{\mathbf{h}}$ is defined on a single q–variable by the formula

$$d^{\mathbf{h}} q_\gamma = \sum_{A,I^-,I^+} \frac{n_{0,0}^A\big(\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+},\gamma\big)}{C(I^-)\, C(I^+)}\, e^A \mathbf{q}^{I^-} \mathbf{p}^{I^+}$$

and on a p–variable respectively by

$$d^{\mathbf{h}} p_\gamma = (-1)^{|p_\gamma|+1} \sum_{A,I^-,I^+} \frac{n^A_{0,0}\big(\gamma, \boldsymbol{\gamma}^{I^-}; \boldsymbol{\gamma}^{I^+}\big)}{C(I^-)\,C(I^+)}\, e^A \mathbf{q}^{I^-} \mathbf{p}^{I^+} \ .$$

We are summing over all combinations of monomials $e^A \mathbf{q}^{I^-} \mathbf{p}^{I^+}$. For the definition of the rational numbers $n^A_{0,0}(\dots)$, we refer to Section "Notation: Closed Reeb orbits and holomorphic curves". For arbitrary elements in $\mathcal{A}_{rSFT}$ extend the operator $d^{\mathbf{h}}$ by using the graded Leibniz rule.

Full SFT. The algebra of symplectic field theory $\mathcal{A}_{SFT}$ is composed of formal power series of the form

$$F = \sum_{g=0}^{\infty} \sum_{I^+} f_{g,I^+}(\mathbf{q})\, \mathbf{p}^{I^+}\, \hbar^g \ ,$$

as above $f_{g,I^+}(\mathbf{q})$ is an element in the contact homology algebra, and $\hbar$ is a new formal variable of degree $2\,(n-3)$, so that the total degree of a monomial $ce^A\, \mathbf{q}^{I^-} \mathbf{p}^{I^+} \hbar^g$ is given by

$$\Big| c\, e^A\, \mathbf{q}^{I^-} \mathbf{p}^{I^+} \hbar^g \Big| = \Big| e^A\, \mathbf{q}^{I^-} \mathbf{p}^{I^+} \Big| + 2g\,(n-3) \ .$$

Unlike $\mathcal{A}_{CH}$ and $\mathcal{A}_{rSFT}$, the algebra $\mathcal{A}_{SFT}$ is not supercommutative, but instead has the following commutator relations. For two q–variables or two p–variables the commutator relations are identical to the ones of rational SFT, but for mixed terms, we require the supersymmetric relation

$$[q_\gamma, p_{\gamma'}] := q_\gamma p_{\gamma'} - (-1)^{|q_\gamma||p_{\gamma'}|} p_{\gamma'} q_\gamma = \begin{cases} \kappa_\gamma \hbar & \text{if } \gamma = \gamma' \\ 0 & \text{otherwise.} \end{cases}$$

One way to incorporate this relation into a formalism is by representing p_γ as the derivation operator $\kappa_\gamma \hbar\, \partial/\partial q_\gamma$. The $\hbar$–variable commutes with the q– and p–variables.

In [EGH00], the differential $d^{\mathbf{H}}$ of SFT was given as the commutator with a distinguished element $\mathbf{H}$, but here we will just specify the effect of $d^{\mathbf{H}}$ on the generators of $\mathcal{A}_{SFT}$

$$d^{\mathbf{H}} \hbar = 0 \ ,$$

$$d^{\mathbf{H}} q_\gamma = \sum_{g,A,I^-,I^+} \frac{n^A_{g,0}\big(\boldsymbol{\gamma}^{I^-}; \boldsymbol{\gamma}^{I^+}, \gamma\big)}{C(I^-)\,C(I^+)}\, e^A \mathbf{q}^{I^-} \mathbf{p}^{I^+} \hbar^g \ ,$$

and

$$d^{\mathbf{H}} p_\gamma = (-1)^{|p_\gamma|+1} \sum_{g,A,I^-,I^+} \frac{n^A_{g,0}\big(\gamma, \boldsymbol{\gamma}^{I^-}; \boldsymbol{\gamma}^{I^+}\big)}{C(I^-)\,C(I^+)}\, e^A \mathbf{q}^{I^-} \mathbf{p}^{I^+} \hbar^g \ ,$$

and extend it to general elements by the graded Leibniz rule.

Marked points. Choose closed differential forms $\Theta^1, \dots, \Theta^d$ that represent an integral basis for the de Rham cohomology ring $H^*_{\mathrm{dR}}(M)$. Take any of the algebras described above, i.e., let $\mathcal{A}$ be either the contact homology algebra $\mathcal{A}_{CH}$, the rational symplectic field algebra $\mathcal{A}_{rSFT}$, or the SFT algebra $\mathcal{A}_{SFT}$. Define new formal variables $t_1, \dots, t_d$ with grading $|t_j| := \deg \Theta^j - 2$. Let $\mathcal{A}^*$ be the algebra of formal power series in the $t_1, \dots, t_d$ with coefficients in $\mathcal{A}$ such that the t_j supercommute among themselves, and with the q–variables, and possibly also (if they are part of $\mathcal{A}$) with the p–, and $\hbar$–variables.

The differential ∂^* on $\mathcal{A}^*$ vanishes on $\hbar$, and the $t_1, \dots, t_d$

$$\partial^* t_j = 0, \text{ and } \partial^* \hbar = 0 \ .$$

To define ∂^* on the q– and p–variables, introduce first the notation $\boldsymbol{\Theta} = \sum_{j=1}^d \Theta^j t_j$, and

$$\langle n^A_{g,m}(\dots) | (\boldsymbol{\Theta}, \dots, \boldsymbol{\Theta}) \rangle = \sum_{1 \le a_1, \dots, a_m \le d} \langle n^A_{g,m}(\dots) | (\Theta^{a_1}, \dots, \Theta^{a_m}) \rangle\, t_{a_1} \cdots t_{a_m} \ ,$$

and then set

$$\partial^* q_\gamma = \sum_{g,m,A,I^-,I^+} \frac{\langle n^A_{g,m}\big(\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+},\gamma\big)|(\boldsymbol{\Theta},\dots,\boldsymbol{\Theta})\rangle}{C(I^-)\,C(I^+)}\, e^A \mathbf{q}^{I^-}\,\mathbf{p}^{I^+}\,\hbar^g \;,$$

and

$$\partial^* p_\gamma = (-1)^{|p_\gamma|+1} \sum_{g,m,A,I^-,I^+} \frac{\langle n^A_{g,m}\big(\gamma,\boldsymbol{\gamma}^{I^-};\boldsymbol{\gamma}^{I^+}\big)|(\boldsymbol{\Theta},\dots,\boldsymbol{\Theta})\rangle}{C(I^-)\,C(I^+)}\, e^A \mathbf{q}^{I^-}\,\mathbf{p}^{I^+}\,\hbar^g \;,$$

where we are summing over all I^- described above, and in case $\mathcal{A} = \mathcal{A}_{CH}$, we assume that $g = 0$, and $I^+ = \mathbf{0}$. If $\mathcal{A} = \mathcal{A}_{rSFT}$, we still keep $g = 0$, but allow any sequence I^+, and finally if $\mathcal{A} = \mathcal{A}_{SFT}$, any integer $g \geq 0$, and sequence I^+ is allowed.

3. Proof of Theorem 1

3.1. **The implications** $(ii) \Rightarrow (i)$**,** $(iii) \Rightarrow (i)$**, and** $(n + iii) \Rightarrow (n)$**.** The statements are based on the following trivial remark.

Remark 3. Let $\pi : \mathcal{A}' \to \mathcal{A}$ be a chain map between two differential graded algebras $(\mathcal{A}, \partial)$ and $(\mathcal{A}', \partial')$. From $\pi \circ \partial' = \partial \circ \pi$ it follows immediately that an exact element $f' \in \mathcal{A}'$ is mapped to an exact element $f = \pi(f') \in \mathcal{A}$.

Corresponding to each of the cases $(ii) \Rightarrow (i)$, $(iii) \Rightarrow (i)$, and $(n + iii) \Rightarrow (n)$, we find an element g in either $\mathcal{A}_{rSFT}$, $\mathcal{A}_{SFT}$, or $\mathcal{A}^*$ such that $d^{\mathbf{h}} g = \mathbf{1}$, $d^{\mathbf{H}} g = \mathbf{1}$, or $\partial^* g = \mathbf{1}$.

For the first case, define a projection $\pi : \mathcal{A}_{rSFT} \to \mathcal{A}_{CH}$ by mapping any monomial

$$f_{I^+}(\mathbf{q})\,\mathbf{p}^{I^+} \mapsto \begin{cases} f_{I^+}(\mathbf{q}) & \text{if } I^+ = \mathbf{0} \\ 0 & \text{otherwise,} \end{cases}$$

and extending this map linearly. It is clear that π will be an algebra homomorphism, and to see that it is a chain map, just compare the definitions of the differentials ∂ and $d^{\mathbf{h}}$. All terms counted by the contact homology differential also appear in the rational SFT differential, and it is clear that $\partial \circ \pi = \pi \circ d^{\mathbf{h}}$ holds, if $d^{\mathbf{h}}$ cannot decrease the number of p–variables in any monomial. Note that already by the Leibniz rule, the differential $d^{\mathbf{h}}$ can decrease the number of factors in a monomial at most by one, and to decrease the number of p–factors, variables p_γ have to exist such that $d^{\mathbf{h}} p_\gamma$ contains (non zero) terms without any p–coordinates at all. This would only be possible, if there were non empty moduli spaces $\mathcal{M}^A_{0,0}\big(\boldsymbol{\gamma}^{\mathbf{0}};\gamma,\boldsymbol{\gamma}^{I^-}\big)$ of spheres without positive punctures, but by the maximum principle no such curves exist. We show in Appendix A that a weaker form of the maximum principle still holds for solutions of perturbed Cauchy-Riemann equations, so that the same conclusion remains true. It follows that $\pi\big(d^{\mathbf{h}} f_{I^+}(\mathbf{q})\,\mathbf{p}^{I^+}\big) = 0$ for any monomial with $I^+ \neq \mathbf{0}$, and so $\pi(g) \in \mathcal{A}_{CH}$ will be a primitive of the unit element $\mathbf{1}$.

The chain map $\pi : \mathcal{A}_{SFT} \to \mathcal{A}_{CH}$ for the second case will be defined similarly, dropping any monomial that contains a positive $\hbar$– or p–power. This map is compatible with the commutator relations, and it is again a chain map, because $d^{\mathbf{H}}$ cannot decrease the number of p–factors without raising the $\hbar$–power and vice versa such that $\pi\big(d^{\mathbf{H}} f_{I^+}(\mathbf{q})\,\mathbf{p}^{I^+}\hbar^g\big) = 0$, if either $g \neq 0$ or $I^+ \neq \mathbf{0}$, and so the contact homology algebra is trivial as we wanted to show.

To prove $(n+iii) \Rightarrow (n)$, let $\pi : \mathcal{A}^* \to \mathcal{A}$ be the projection that drops any monomial containing a t_j–variable. Let ∂ be the differential of $\mathcal{A}$ (i.e., depending on $\mathcal{A}$ either ∂, $d^{\mathbf{h}}$ or $d^{\mathbf{H}}$). We need to show π is a chain map. As before the argument here is that ∂ is the zero order term of ∂^* in the t_j, and this is true because the count $n^A_g(\dots)$ coincides by definition with $\langle n^A_{g,0}(\dots)|()\rangle$, furthermore ∂^* can never decrease the number of t_j–variables of a monomial, so that $\partial \pi f = \mathbf{1}$.

3.2. **The implications** $(i) \Rightarrow (ii)$**,** $(i) \Rightarrow (iii)$**, and** $(n) \Rightarrow (n{+}iii)$**.** For the implication $(i) \Rightarrow (ii)$, assume that f_0 is an element in the contact homology algebra $\mathcal{A}_{CH}$ such that $\partial f_0 = \mathbf{1}$. This f_0 canonically embeds into the rational SFT algebra $\mathcal{A}_{rSFT}$, where we can compute its differential

$d^{\mathbf{h}} f_0 = \mathbf{1} - g$. The element $g \in \mathcal{A}_{rSFT}$ is always closed, because $d^{\mathbf{h}} g = d^{\mathbf{h}}(\mathbf{1} - d^{\mathbf{h}} f) = 0$, and all of its terms contain at least one p–variable. The formal inverse of $\mathbf{1} - g$ is given by

$$(\mathbf{1} - g)^{-1} := \sum_{k=0}^{\infty} g^k \ ,$$

where we set $g^0 = \mathbf{1}$. This object is well defined in $\mathcal{A}_{rSFT}$, because all terms in g have at least one p–factor, so that only the powers of g up to $k = |I^+|$ contribute to the terms $\mathbf{p}^{I^+}$ in $(1-g)^{-1}$, and in particular $(\mathbf{1}-g)^{-1}$ is a formal power series in the p–variables. It is obvious that $d^{\mathbf{h}}(1-g)^{-1} = 0$.

Define an element $f \in \mathcal{A}_{rSFT}$ by

$$f := f_0\, (\mathbf{1} - g)^{-1} \ .$$

It easily follows that

$$d^{\mathbf{h}} f = d^{\mathbf{h}} \left(f_0\, (\mathbf{1} - g)^{-1}\right) = (d^{\mathbf{h}} f_0)\, (\mathbf{1} - g)^{-1} - f_0\, d^{\mathbf{h}} (1 - g)^{-1} = (\mathbf{1} - g)\, (\mathbf{1} - g)^{-1} = \mathbf{1} \ ,$$

just as we wanted to show.

We will now prove the implication $(i) \Rightarrow (iii)$ in a similar way: Assume that $f \in \mathcal{A}_{CH}$ is such that $\partial f = \mathbf{1}$. We can canonically embed the contact homology algebra $\mathcal{A}_{CH}$ into the algebra of symplectic field theory $\mathcal{A}_{SFT}$, since the commutation relations for only q–variables are identical in both spaces. As we said above, dropping any term with a p– or $\hbar$–variable the differential $d^{\mathbf{H}}$ coincides on $\mathcal{A}_{CH}$ with the differential ∂ of contact homology. Thus

$$d^{\mathbf{H}} f = \partial f - G = \mathbf{1} - G \ ,$$

where G only contains monomials with a non zero power of $\hbar$ or of p, therefore the formal inverse $(\mathbf{1} - G)^{-1} = \sum_{k=0}^{\infty} G^k$ is a well defined element in $\mathcal{A}_{SFT}$. Moreover it is closed, and so

$$F = f \sum_{k=0}^{\infty} G^k$$

is a primitive of $\mathbf{1}$, because by using the Leibniz rule

$$d^{\mathbf{H}} \left(f \sum_{k=0}^{\infty} G^k \right) = \left(d^{\mathbf{H}} f\right) \sum_{k=0}^{\infty} G^k - f \sum_{k=0}^{\infty} d^{\mathbf{H}} \left(G^k\right) = \mathbf{1} \ ,$$

follows, and the homology of SFT vanishes.

Finally, we compare the invariants with and without marked points, and prove $(n) \Rightarrow (n + iii)$: Let thus $\mathcal{A}$ be either $\mathcal{A}_{CH}$, $\mathcal{A}_{rSFT}$, or $\mathcal{A}_{SFT}$ without marked points, and $\mathcal{A}^*$ the corresponding algebra with marked points. Use that ∂ and ∂^* are identical in 0–th order of t_j–powers, so that if $\partial f = \mathbf{1}$, we have that $\partial^* f = \mathbf{1} - G$, where all terms in G have positive t_j–powers. As above, the formal inverse $(1 - G)^{-1} = \sum G^k$ is an element of $\mathcal{A}^*$, so that we can define $F := f\,(1 - G)^{-1}$ which is a primitive of $\mathbf{1}$ with respect to ∂^*.

Appendix A. Maximum principle for perturbed holomorphic curves

Let $T_0 > 0$ be smaller than the period of any closed Reeb orbit in M. Let $\widehat{G}$ be the positive (but not definite) metric on $\mathbb{R} \times M$ defined by $\widehat{G}(\cdot, \cdot) = d\alpha(\cdot, J\cdot)$. Let G be the positive definite metric on $\mathbb{R} \times M$ such that the Reeb field R_α and the Liouville field $\frac{\partial}{\partial t}$ are mutually orthogonal, are orthogonal to ξ, and have unit length. In the next proposition, $\nu \in \Lambda^{0,1}(\Sigma_g, \tilde{u}^* T(\mathbb{R} \times M))$ will denote a perturbation for the Cauchy-Riemann equation.

Proposition 3. *A curve $\tilde{u} : \Sigma_g \setminus (\mathbf{x} \cup \mathbf{z}) \to \mathbb{R} \times M$ that satisfies the perturbed Cauchy-Riemann equation*

$$d\tilde{u} + J \circ d\tilde{u} \circ j = \nu$$

with $\|\nu\|_{L^2(G)} < 2\sqrt{T_0}$ has to have top punctures $\mathbf{x} \neq \emptyset$.

Proof. Assume that there exists a map $\tilde{u} : \Sigma_g \setminus \mathbf{z} \to \mathbb{R} \times M$ satisfying $d\tilde{u} + J \circ d\tilde{u} \circ j = \nu$ that is asymptotic for $t \to -\infty$ to the orbits $\gamma_1, \dots, \gamma_s$. By Stokes theorem, we have

$$\int_{\Sigma_g \setminus \mathbf{z}} \tilde{u}^* d\alpha = -\sum_{i=1}^{s} T_i < -T_0 \ ,$$

where T_i is the period of γ_i.

On the other hand, let $z = x + iy$ be coordinates of a complex chart on Σ_g such that $\|\partial_x\| = \|\partial_y\| = 1$. Then

$$d\alpha\big(\partial_x \tilde{u}, \partial_y \tilde{u}\big) = -d\alpha\big(J\partial_y \tilde{u}, \partial_y \tilde{u}\big) + d\alpha\big(\nu(\partial_x), \partial_y \tilde{u}\big) = \widehat{G}\big(\partial_y \tilde{u}, \partial_y \tilde{u}\big) + \widehat{G}\big(\nu(\partial_x), -J\,\partial_y \tilde{u}\big) \ .$$

After a suitable rotation in the (x, y)–plane, we obtain, using Cauchy-Schwarz inequality, and $\big(\|d\tilde{u}\|_{\widehat{G}} - 1/2\,\|\nu\|_{\widehat{G}}\big)^2 \geq 0$

$$d\alpha\big(\partial_x \tilde{u}, \partial_y \tilde{u}\big) = \|d\tilde{u}\|^2_{\widehat{G}} + \widehat{G}\big(\nu(\partial_x), -J\,\partial_y \tilde{u}\big) \geq \|d\tilde{u}\|^2_{\widehat{G}} - \|\nu\|_{\widehat{G}}\,\|d\tilde{u}\|_{\widehat{G}} \geq -\frac{1}{4}\,\|\nu\|^2_{\widehat{G}} \ ,$$

where $\|\cdot\|_{\widehat{G}}$ is the semi-norm induced by $\widehat{G}$.

Integrating over $\Sigma_g \setminus \mathbf{z}$, we obtain

$$\int_{\Sigma_g \setminus \mathbf{z}} \tilde{u}^* d\alpha \geq -\frac{1}{4}\,\|\nu\|^2_{L^2(\widehat{G})} \ .$$

Comparing with the Stokes bound for this integral, we obtain $\|\nu\|_{L^2(G)} \geq \|\nu\|_{L^2(\widehat{G})} > 2\sqrt{T_0}$, a contradiction. $\square$

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